# Connecting Limit Ideas

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# Limits in the curriculum

- The "spiralling back" curriculum
  - Circle properties (GCSE)
    - Probably not
  - Graphing asymptotes (GCSE/Early A level)
  - Geometric progressions (GCSE/Early A level)
  - Limits to 'motivate' the tangent (Early-mid A level)
  - Limits as a topic (Late A level - "further pure")
  - Definitions and proofs of limits (Early degree)
    - Analysis as foundations of calculus
  - Limits of functions of many variables (Years 1 and 2 of degree)

# We know all about how students understand limits

- Well over 600 research papers on the limit concepts
- Classical theories developed in the context of limit
  - Concept image/concept definition
  - APOS
  - Epistemological obstacle

# What do we know about students' understanding of limits?

- They're hard
  - The notion of infinity
    - As a "big number" or a process [static/dynamic]
  - The Leibniz assumption
  - Complex definitions
- Even the founders found it hard
  - "Ghosts of departed quantities"

# There's a hole in the literature (dear Liza, dear Liza)

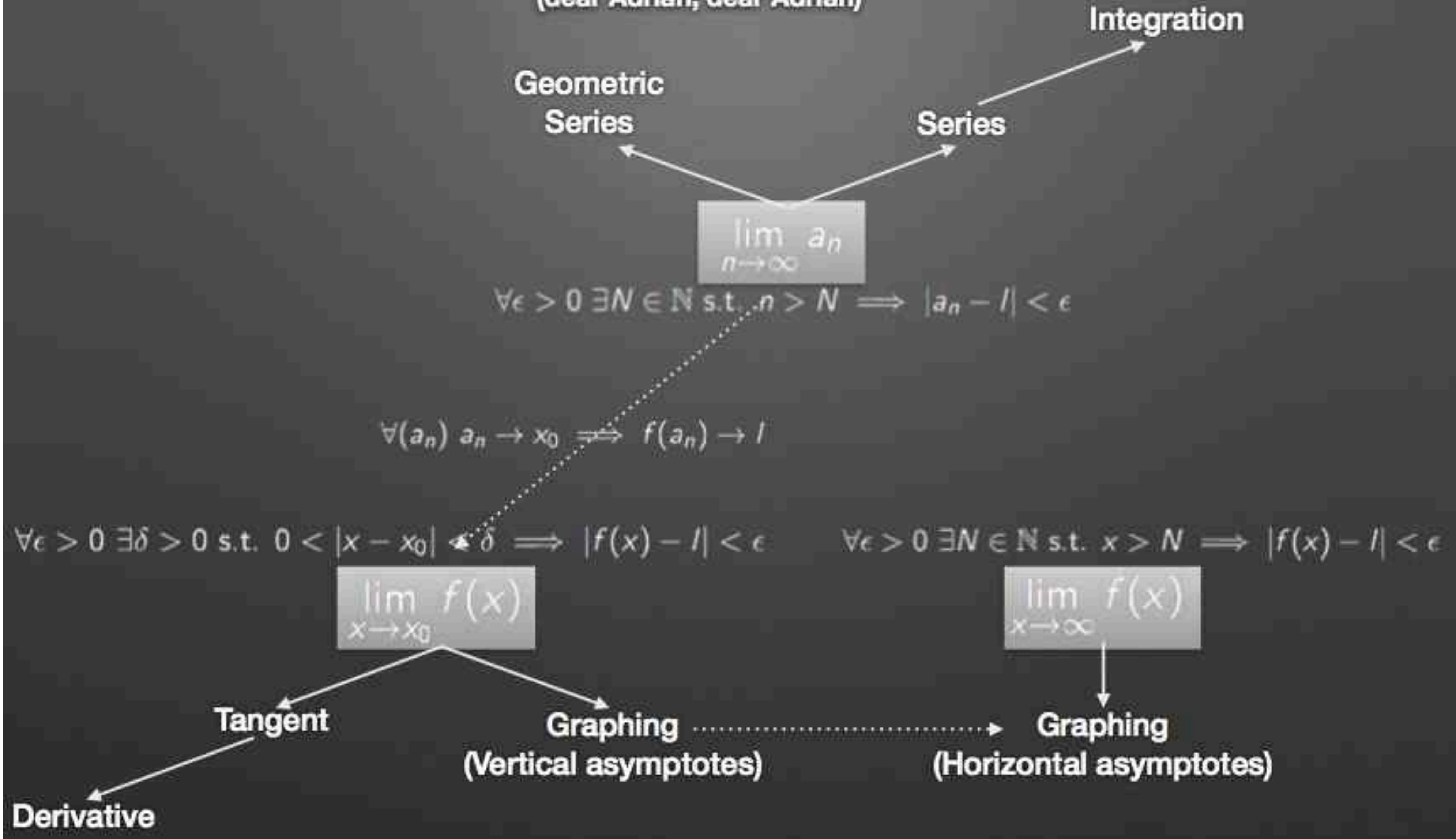
- Much literature deals with a single type of limit
  - Limits of sequences (Sierpinska 1987; Alcock and Simpson 2005; Roh 2008 ...)
  - Limits of functions at a point (Williams 1991; Cottrill et al. 1996; Swinyard 2011...)
  - Limits of functions at infinity (Kidron 2011)
- Some literature looks at a number of different types of limit
  - Tall and Vinner (1981); Monaghan (1991); Elia et al (2009) ...
- But (almost) everyone behaves as if limit is a single concept
  - Przenioslo (2010) - but no-one understands this paper (including Przenioslo!)
- No-one's asked the students if it is

# There's a hole in the literature (dear Liza, dear Liza)

- Worse - some people explicitly say students' understandings are connected with no empirical evidence
- Cottrill et al (1996)
  - Function at a point and sequences within the genetic decomposition
- Lakoff and Núñez (2000)
  - Limit of a function at a point necessarily conceived as co-ordination of sequences
  - Basic metaphor of infinity - *must* be dynamic
    - Non-standard analysis says not

# So what does it look like?

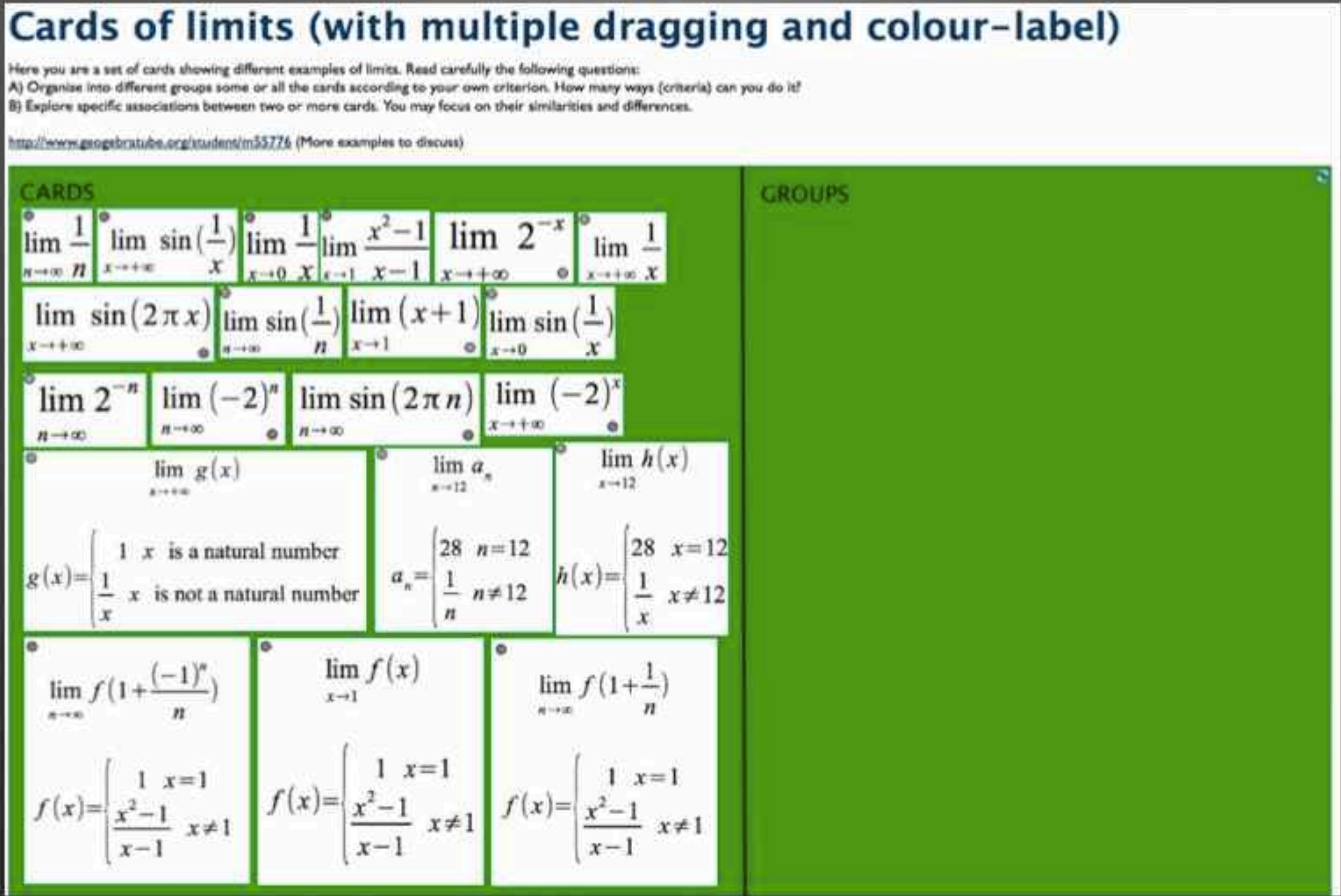
(dear Adrian, dear Adrian)



# So how will we fill it? (dear José, dear José)

- Card sorting tasks
  - Repeated single criterion open-sort
  - Closed card comparisons
- 1st and 2nd year mathematics students
- 40 minutes - 1 hour
- “Think aloud”

# So how will we fill it? (dear José, dear José)



## Cards of limits (with multiple dragging and colour-label)

Here you are a set of cards showing different examples of limits. Read carefully the following questions:

A) Organise into different groups some or all the cards according to your own criterion. How many ways (criteria) can you do it?

B) Explore specific associations between two or more cards. You may focus on their similarities and differences.

http://www.geogebratube.org/student/m55776 (More examples to discuss)

**CARDS**

$\lim_{n\to\infty}\frac{1}{n}$ | $\lim_{x\to+\infty}\sin(\frac{1}{x})$ | $\lim_{x\to0}\frac{1}{x}$ | $\lim_{x\to1}\frac{x^2-1}{x-1}$ | $\lim_{x\to+\infty}2^{-x}$ | $\lim_{x\to+\infty}\frac{1}{x}$

$\lim_{x\to+\infty}\sin(2\pi x)$ | $\lim_{n\to\infty}\sin(\frac{1}{n})$ | $\lim_{x\to1}(x+1)$ | $\lim_{x\to0}\sin(\frac{1}{x})$

$\lim_{n\to\infty}2^{-n}$ | $\lim_{n\to\infty}(-2)^n$ | $\lim_{n\to\infty}\sin(2\pi n)$ | $\lim_{x\to+\infty}(-2)^x$

$\lim_{x\to+\infty}g(x)$, $g(x)=\begin{cases}1 & x \text{ is a natural number}\\ \frac{1}{x} & x \text{ is not a natural number}\end{cases}$

$\lim_{n\to12}a_n$, $a_n=\begin{cases}28 & n=12\\ \frac{1}{n} & n\neq12\end{cases}$

$\lim_{x\to12}h(x)$, $h(x)=\begin{cases}28 & x=12\\ \frac{1}{x} & x\neq12\end{cases}$

$\lim_{n\to\infty}f(1+\frac{(-1)^n}{n})$, $f(x)=\begin{cases}1 & x=1\\ \frac{x^2-1}{x-1} & x\neq1\end{cases}$

$\lim_{x\to1}f(x)$, $f(x)=\begin{cases}1 & x=1\\ \frac{x^2-1}{x-1} & x\neq1\end{cases}$

$\lim_{n\to\infty}f(1+\frac{1}{n})$, $f(x)=\begin{cases}1 & x=1\\ \frac{x^2-1}{x-1} & x\neq1\end{cases}$

**GROUPS**

# Results

- Preliminary categories related to open sorting cards criteria.
- Individual choices of cards to compare
- Results from discussion on cards researcher suggested to compare

# Preliminary Categories

- Core concepts are not involved
  - A.1. Focus on formulae in a indeterminate variable
    - Families of functions
    - Format (Single formula/ definition by conditions)
    - Occurrence of specific symbols or expressions
  - A.2. Focus on the independent variable (Functions – Sequences)
  - A.3. Properties of fucntions (Range, critical points, etc.)

# Preliminary Categories

- Core concepts are involved
  - B.1. Focus on the character (finite or infinite) of the x,n-value at which the limit is studied.
    - Focus on the specific x, n- value
  - B.2. Existence of the limit
    - Focus on the specific limit value regardless the variable x or n.
    - Focus on the specific limit value and the variable x or n.
  - B.3. Behaviour of the function near the point at which the limit is studied.

# Preliminary Categories

- B. Core concepts are involved
  - B.3. Behaviour of the function near the point at which the limit is studied
    - Sign of the function
    - Monotonic convergence to the limit at infinity/ oscillating behaviour
    - Kinds of discontinuity and continuity at x-point or infinity or at the domain

# The Cards

$$\lim_{n\to\infty}\frac{1}{n}$$

$$\lim_{x\to+\infty}\sin\left(\frac{1}{x}\right)$$

$$\lim_{x\to0}\frac{1}{x}$$

$$\lim_{x\to1}\frac{x^2-1}{x-1}$$

$$\lim_{x\to+\infty}2^{-x}$$

$$\lim_{x\to+\infty}\frac{1}{x}$$

$$\lim_{x\to+\infty}\sin(2\pi x)$$

$$\lim_{n\to\infty}\sin\left(\frac{1}{n}\right)$$

$$\lim_{x\to1}(x+1)$$

$$\lim_{x\to0}\sin\left(\frac{1}{x}\right)$$

$$\lim_{n\to\infty}2^{-n}$$

$$\lim_{n\to\infty}(-2)^n$$

$$\lim_{n\to\infty}\sin(2\pi n)$$

$$\lim_{x\to+\infty}(-2)^x$$

$$\lim_{x\to+\infty}g(x)$$

$$g(x)=\begin{cases}1 & x \text{ is a natural number}\\ \frac{1}{x} & x \text{ is not a natural number}\end{cases}$$

$$\lim_{n\to12}a_n$$

$$a_n=\begin{cases}28 & n=12\\ \frac{1}{n} & n\neq12\end{cases}$$

$$\lim_{x\to12}h(x)$$

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$$\lim_{n\to\infty}f\left(1+\frac{(-1)^n}{n}\right)$$

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$$\lim_{x\to1}f(x)$$

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$$\lim_{n\to\infty}f\left(1+\frac{1}{n}\right)$$

$$f(x)=\begin{cases}1 & x=1\\ \frac{x^2-1}{x-1} & x\neq1\end{cases}$$

# Symmetrical behaviour of $\frac{1}{x}$ at $x = 0$ and at infinity

Symmetrical behaviour of 1/x at x=0 and at infinity.

# Choice: “1 over something increasing has limit 0”

Choice: "1 over something increasing has limit 0"

# Suggestion from researcher

Suggestion from researcher:

Compare $\lim f(x)$ at $x=1$

$\lim f(1+1/n)$ and

$\lim f(1+(-1)^n/n)$

# Relationship between limit of functions at infinity and limit of sequences

Relationship between limit of functions at infinity and limit of sequences

# Conclusions

- *x* and *n* are at the beginning mostly interpreted as general “real variables”
- The used criteria to organise card cover nearly all the possible elements cards include.

# Conclusions

- With regard to individual choices to compare cards the most relevant relationship between core concepts are the following ones:
- Relationship of symmetry between limits at x=0 and at infinity by the inverse transformation $\frac{1}{x}$.
- Existence of $\lim_{n\to\infty} f(n)$ is necessary, but not sufficient for the existence of $\lim_{x\to+\infty} f(x)$

# Conclusions

- Existence of $a_n \to x_0, \lim_{n\to\infty} a_n = L$ is necessary but not sufficient for the existence of $\lim_{x\to x_0} f(x) = L$