12th International Congress on Mathematical Education
Topic Study Group 13
8 July – 15 July, 2012, COEX, Seoul, Korea

MEANINGS OF THE CONCEPT OF FINITE LIMIT OF A FUNCTION AT ONE POINT, AS DEMONSTRATED BY STUDENTS IN BACHILLERATO

Fernández Plaza, José Antonio; Ruiz Hidalgo, Juan Francisco; Rico Romero Luis

University of Granada (Spain)

joseanfplaza@ugr.es; jfruiz@ugr.es; lrico@ugr.es

In this paper, we present some results of an exploratory study performed with students of ages 16-17. We investigate the different uses that these students make of terms such as "to approach", "to tend toward", "to reach", "to exceed" and "limit", terms that describe some properties of the concept of the finite limit of a function at a point. We use the interpretive framework of conceptual analysis to infer the meanings that students associate with the main terms used in their answers.

Keywords: Finite limit of a function at a point, conceptual analysis, meanings and verbal arguments, non-compulsory secondary education.

INTRODUCTION

This paper presents an exploratory, descriptive study that focuses on the meanings that Spanish students in Bachillerato[1] (16-17 years old) associate with the concept of the finite limit of a function at a point.

We base our study on prior research on cognitive conflicts connected to the concepts of real number limit, notion of infinity, and continuity of a function (Tall & Vinner, 1981; Davis & Vinner, 1986; Monaghan, 1991; Cornu, 1991).

In contrast to the everyday meanings, we analyze conceptually the mathematical meaning and use of the key terms, an analysis useful for interpreting the conception that the subjects have of the concept of finite limit of a function at a point.

RESEARCH PROBLEM

We propose to describe:

- how students express verbally their intuitive conceptions of the notion of finite limit of a function at a point, and
- how students interpret and respond to tasks connected to this concept by analyzing the meaning of key terms that express different facets of the concept of limit.

[1] Non-compulsory secondary education

THEORETICAL FRAMEWORK AND PRIOR RESEARCH

We position this study in the research agenda of *Advanced Mathematical Thinking*, in the international group on the Psychology of Mathematics Education (Gutiérrez & Boero, 2006, pp. 147-172). We assume the difficulty of defining the transition from elementary to advanced mathematical thinking.

Azcárate and Camacho (2003) underscore the importance of the definitions for advanced mathematics as a characteristic that distinguishes elementary from advanced mathematics, grounding the description in sufficient experience in elementary mathematics. The educational stage analyzed assumes a period of transition in which students use elementary techniques to tackle mathematics contents whose historical, epistemological, and didactic development have advanced status.

We assume the notion of the meaning of a mathematical concept developed by Rico (2007). We analyze the systems of representation, formal aspects or references of the concept, and the phenomena that give it meaning.

Prior research

Monaghan (1991) studies the influence of language on the ideas that students have about the terms “to tend toward”, “to approach”, “to converge at”, and “limit”, as these terms are employed with different graphs of functions and the examples that school students use. We recognize as a limitation the approach adopted in this case, in which the key terms that the students were to use were defined a priori, instead of enabling students to use their own words freely and spontaneously and to infer the appropriate nuances a posteriori.

STUDY DESIGN

We describe the conceptual analysis of key terms designed to establish their mathematical use and to contrast this employment with their colloquial use or use in other disciplines, according to the different conceptions of these terms expressed by students.

Conceptual analysis of key terms

We analyze the terms “approach”, “tend toward”, “reach”, “exceed” and “limit”. We chose these terms for the following reasons:

- They are terms with a technical and formal meaning in mathematics, but they also have ordinary colloquial uses not connected to their mathematical meanings.
- They appear frequently in the literature, both in the definition of the concept of limit and in the characterization of the associated difficulties and errors; they show conflicts between formal and colloquial uses.
- The subjects in this study use these terms, as well as synonyms, to express different interpretations of the concept of limit, both technically (terminology acquired through instruction mediated by the professor, the textbook, or their own instrument for data collection) and informally (their own personal interpretations).

- Each of the terms refers in part to properties and modes of usage associated with the concept of limit.

Conceptual analysis identifies the mathematical use of these terms as different from their everyday meanings; it is thus necessary to interpret the conception that the subjects hold.

We follow the dictionaries of the Spanish Royal Academy (Real Academia Española [RAE]) (2001), the Spanish Royal Academy of Science (Real Academia de las Ciencias [RAC]) (1990) and the Oxford Dictionary (2011) to establish the accepted, common, and mathematical meanings of the following terms in Spanish: "to approach", "to tend toward", "to exceed", "to reach" and "limit".

"To tend toward" means "to approach gradually but never reach the value" (RAE, 2001) and expresses a very specific form of approach. Blázquez, Gatica and Ortega (2009) argue that a sequence of numbers approaches a number if the error decreases gradually, but they argue that a sequence "tends toward a limit" if any approach to the limit can be measured by the terms in the sequence. We establish a distinction between these two terms.

A study by Monaghan (1991) concludes that many students do not distinguish between "tend toward" and "approach" in a mathematical context.

The correct use of the term "to tend toward" should be determined using the variable x and not f(x), since the expression "*f(x)* tends toward *L,* when *x* tends toward *a*" can cause cognitive conflicts, as Tall & Vinner (1981) note.

"To reach" is intuitively "to arrive at" or "to come to touch" (RAE, 2001; Oxford, 2011). We interpret "reach" as meaning that a function *reaches the limit* if the limit value is the image of the point at which the limit is studied (continuity); by extension, the limit can be the value of any other point in the domain.

We see that "to exceed" means colloquially "to be above an upper level" (RAE, 2001), excluding the meaning "to be below a lower level". We will say that *the limit of a function may be exceeded* if we can construct two successive monotones of images that converge at the limit, one ascending and the other descending, for appropriate sequences of values of x that converge at the point at which the limit is studied. The reachability or exceedability of the finite limit of a function at a point is not global but local, and there is no logical implication of the two concepts.

The term "limit" has colloquial meanings that interfere with students' conceptions of this term, such as ideas of ending, boundary, and what cannot be exceeded (RAE, Oxford, op. cit). The term's scientific-technical use is related in some disciplines to a subject matter or extreme state in which the behavior of specific systems changes abruptly (RAC, 1990).

Instrument

We worked with a questionnaire containing three open-response questions and one question with closed response. Two different versions of the questionnaire were used, A and B, which may be consulted in Appendix I (Fernández-Plaza, 2011).

This report presents the open items, which require that the respondent evaluate as true (T) or False (F) the statement of a property related to the concept of the limit of a function at a point and then to justify the option chosen. The following shows one of the items as an example:

1. Circle T or F for each of the following statements, depending on whether it is true or false. Use the box to explain your choice: (Adapt. and trans., Lauten et al., 1994, p. 229)

b) A limit is a number or point that a function cannot exceed.

Table 1 summarizes the data from the six open items proposed:

Table 1: Key terms of the true/false items in the questionnaire

Questionnaire/task	Item code	Key term(s) in the statement
A, task a)	A1.1.a	Motion of the function
A, task b)	A1.1.b	Does not exceed
A, task c)	A1.1.c	Test values and reach
B, task a)	B1.1.a	Approach but not reach
B, task b)	B1.1.b	Approach as precisely as one wishes
B, task c)	B1.1.c	Approach arbitrarily

Sample

The sample was composed of 36 Spanish students in the first year of non-compulsory secondary school study (Bachillerato), 16-17 years of age, who were taking the subject of Mathematics. The students were chosen deliberately and based on their availability. The test was performed halfway through the 2010-2011 school year.

18 subjects answered version A, and another 18 subjects answered version B. The test was performed during a regular session of their mathematics class.

RESULTS

We provide an example of the analysis using the task described above. This analysis was performed in two phases, of which we will describe the first. The second phase consisted of characterizing the categories of response.

Use and counting of the key terms in the written records

We identified and tabulated the different uses of the key terms in the written records provided by the students, without making inferences from their meaning. The groupings of key terms were developed from the review described above. Since we did not require the students to define the key terms, we focus on the presence/absence of these terms and on the use the students make of the terms as they articulate their decisions.

In order to clarify how the tabulation was done, we show three examples provided by the students belonging to the different groups. Firstly, a sample answer from Reachability group, where the underlined expression is related to an use of the key term *to approach/ not reach* is: "A limit is a point to which a function approaches infinitely without reaching it"., Secondly, the answer "A function can indeed surpass a limit, since in many cases to find out the limit we have to give x-values that correspond to bigger images" is a sample within Exceedability group with key term *surpass*. Finally, another sample answer such as "A limit is a number or a point to which a function approaches but it never gets to touch or exceed it" belongs to Mixed group due to the use of the key term *approach/not touch/not exceed*.

Table 2 shows the frequency of some key terms connected to reachability and/or exceedability to characterize the value of the limit, as well as the descriptions of the process of convergence through terms such as *reach* or *approach*. We consider three natural groups (reachability, exceedability, and mixed). Further information, see Fernández-Plaza (2011, p.40, table 4.3.)

Table 2: Responses and frequencies of use of the key terms

Groups	Key terms	No. of responses
Reachability	Reach	1(Affirm.) / 2 (Neg.[2])
	Approach/not reach	2
Exceedability	Exceed	1
	Surpass	1
Mixed	Approach/not touch/not exceed	1
Others/No answer		10
Total		18

The references to reachability predominate (6 of 8 valid) when the subjects are required to argue about the exceedability (2 of 8 valid). This shows a connection between the two properties. We can speculate that this relationship is suggested by the imprecise use of examples, in which convergence is strictly monotone and the value of the limit is, in fact, an upper level and thus unreachable, thereby excluding from the student's reasoning the image of the point at which the study is made, even when that point coincides with the limit.

[2] Affirm. and neg. mean that there are three isolated uses of the key term "reach"; one of them in a affirmative form, for example, " A limit is a number or point which the function reaches"; and two of them in a negative form (that is to say, not reach), for example, "A limit does not reach the point"

CONCLUSIONS

Conceptual analysis permits us to recognize possible conceptions that arise from the colloquial and everyday use of key terms, uses that induce errors in students' understanding of the concept of finite limit of a function at a point. The conceptual analysis helps us to interpret these responses.

Students use relatively undeveloped and imprecise language, characterized by the use of the terminology provided by the items, as well as some original synonyms. The characterization of the limit as not exceedable or not reachable persists, confirming the influence of colloquial and informal use of the term limit in the students' conceptions indicated by Cornu (1991).

The unreachability of the limit is considered to be the main cause of its unexceedability, and the specific character of exceedability and reachability are deduced through the use of examples. Although we have detected the use of expressions similar to "f(x) tends toward a number, when x tends toward...", we do not find indications to verify the existence of the semantic conflicts reported (Tall & Vinner, 1981; Blázquez, Gatica & Ortega, 2009).

Acknowledgements

This study was performed with aid and financing from Fellowship FPU AP2010-0906 (MEC-FEDER), project EDU2009-11337 of the National Plan for R&D&R (MICIN), Subprogram EDUC, and group FQM-193 of the 3rd Andalusian Research Plan (PAIDI).

References

Azcárate, C. & Camacho, M. (2003) Sobre la Investigación en Didáctica del Análisis. *Boletín de la Asociación Matemática Venezolana*, Vol. X, No. 2, pp. 135-149.

Blázquez, S., Gatica, N. & Ortega, T. (2009). Análisis de diversas conceptualizaciones de límite funcional. *Gaceta de la Real Sociedad Matemática Española (RSME)*, *12*(1), 145-168.

Cornu, B. (1991). Limits. In Tall, D. (Ed.) *Advanced Mathematical Thinking*. (pp.153-166). Netherlands: Kluwer Academic Publishers.

Davis, R.B. & Vinner, S. (1986). The Notion of Limit: Some Seemingly Unavoidable Misconceptions Stages. *Journal of Mathematical Behavior*, 5, 281-303.

Fernández-Plaza, J.A. (2011). *Significados puestos de manifiesto por estudiantes de bachillerato respecto al concepto de límite finito de una función en un punto. Un estudio exploratorio.* Postgraduate Study. Granada: Universidad de Granada. http://fqm193.ugr.es/media/grupos/FQM193/cms/Fernandez%20Plaza_TrabInvTut.pdf.

Gutiérrez, A. & Boero, P. (eds.) (2006): *Handbook of research on the psychology of mathematics education.* Sense Publishers: Rotterdam, Holland.

Lauten, A. D., Graham, K. & Ferrini-Mundy, J. (1994). Student understanding of basic calculus concepts: Interaction with the graphics calculator. *The Journal of Mathematical Behavior*, *13*(2), 225-237.

Monaghan, J. (1991). Problems with the Language of Limits. *For the Learning of Mathematics*, *11*(3), 20-24.

Oxford (2011). *Oxford Dictionaries*. Oxford: Oxford University Press. May be accessed at: http://oxforddictionaries.com. Consulted [24-06-2011].

Real Academia de las Ciencias (1990). *Vocabulario científico y técnico.* Madrid: Espasa- Calpe.

Real Academia Española (2001). *Diccionario de la Lengua Española.*(22nd ed). Madrid. May be accessed at http://www.rae.es/rae.html.Consulted [24-06-2011].

Rico, L. (2007). *Sistema de Significados de un Concepto en las Matemáticas Escolares*. Unpublished document. Granada: Universidad de Granada.

Tall, D. O & Vinner, S. (1981). Concept image and concept definition in mathematics, with special reference to limits and continuity, *Educational Studies in Mathematics*, *12*, 151-169.